QUESTION ONE *(6 Marks)*

MULTIPLE CHOICE: Write the correct alternative on your writing paper.

1. Which of the following is an expression for $\int xe^{2x}dx$? **1**

(A) $e^{2x}\left(\frac{x}{2}-\frac{1}{4}\right)+c$

(B) $e^{2x}\left(\frac{x}{2}-1\right)+c$

(C) $e^{2x}\left(x-\frac{1}{4}\right)+c$

(D) $e^{2x}\left(2x-1\right)+c$

2. Which of the following is the primitive function of $\frac{\cos\sqrt{x}}{\sqrt{x}}$? **1**

(A) $\sin\sqrt{x}+c$

(B) $-\sin\sqrt{x}+c$

(C) $\frac{1}{2}\cos^2\sqrt{x}+c$

(D) $2\sin\sqrt{x}+c$

3. Which of the following is the primitive function of $\frac{1}{\sqrt{4-9x^2}}$? **1**

(A) $\frac{1}{6}\sin^{-1}\frac{3x}{2}+c$

(B) $\frac{1}{3}\sin^{-1}\frac{3x}{2}+c$

(C) $\frac{2}{3}\sin^{-1}\frac{3x}{2}+c$

(D) $\frac{1}{3}\sin^{-1}\frac{2x}{3}+c$

4. Which of the following is equal to $\int_{-2}^{2} \sqrt{4-x^2}$? **1**

(A) 0

(B) π

(C) 2π

(D) 4π

5. Which of the following is the primitive function of $\int \frac{1-\ln x}{x^2}$? **1**

(A) $\frac{\ln x}{x}+c$

(B) $x\ln x+c$

(C) $\ln x^2+c$

(D) $\frac{x}{\ln x}+c$

6. Which of the following statements are true ? **1**

1. $\int_{-\pi}^{\pi} \sin^7\theta = 0$

2. $\int_{-\pi}^{\pi} \cos^7\theta = 0$

(A) Both are true.

(B) Both are false.

(C) Only 1 is true.

(D) Only 2 is true.

<u>QUESTION TWO</u> *(26Marks)* **Start a new page**

(a) Evaluate $\int_0^1 \frac{dx}{\sqrt{4x^2+36}}$. **2**

(b) Find $\int \frac{\cos\theta}{\sin^5\theta}\,d\theta$. **1**

(c) Using the substitution $t=\tan\frac{x}{2}$, evaluate $\int_0^{\frac{\pi}{2}} \frac{d\theta}{3\sin\theta+4\cos\theta}$. **3**

(d) Find $\int \frac{1}{x^2-12x+61}\,dx$. **2**

(e) Use the substitution $x=3\sin\theta$, to evaluate $\int_0^{\frac{3}{\sqrt{2}}} \frac{dx}{(9-x^2)^{\frac{3}{2}}}$. **4**

(f) (i) Find the real numbers a and b such that **2**

$$\frac{x^2-7x+4}{(x+1)(x-1)^2}=\frac{a}{x+1}+\frac{b}{x-1}-\frac{1}{(x-1)^2}.$$

(ii) Hence find $\int \frac{x^2-7x+4}{(x+1)(x-1)^2}\,dx.$ **2**

(g) Find $\int \sqrt{\frac{x+1}{x-1}}\,dx$. **2**

(h) (i) Let $I_n=\int (ln\,x)^n dx$, show that $I_n=x(ln\,x)^n-nI_{n-1}$. **5**

(ii) Hence evaluate $\int_1^{e^4} (ln\,x)^3\,dx$.

(i) Find $\int 2\sin\theta\cos\theta(3\sin\theta-4\sin^3\theta)\,d\theta$ **3**

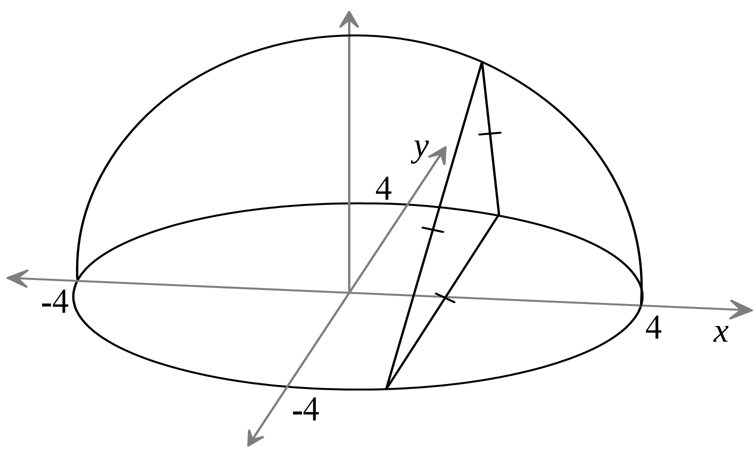
QUESTION THREE *(12 Marks)* **Start a new page**

(a) The region enclosed by the curve $y = 5x - x^2$, the x axis and the lines $x = 1$ and $x = 3$ is rotated about the y axis. By using the method of cylindrical shells, find the volume of the solid so produced. **4**

(b) (i) Sketch the region bounded by the curve $y = \log_e x$, the x-axis and the vertical line $x = e$. 4

(ii) The region is rotated about the y-axis to form a solid. Find the volume of the solid by slicing perpendicular to the axis of rotation.

(c) **4**



The diagram above shows a solid which has the circle $x^2 + y^2 = 16$ as its base. The cross-section perpendicular to the x axis is an equilateral triangle. Calculate the volume of the solid.

END OF PAPER

Question 1

1. $|z| = 2\sqrt{5}$ $\arg z = -\pi + \tan^{-1} 2$

z lies in the 3rd quadrant

$\therefore z = -2-4i$ (C)

2. Let $z^3 = -8$

$z^3 + 8 = 0$

$(z+2)(z^2-2z+4) = 0$

$(z+2)[(z-1)^2+3] = 0$

$(z+2)(z-1+\sqrt{3}i)(z-1-\sqrt{3}i) = 0$

$\therefore z = -2,\ 1-\sqrt{3}i,\ 1+\sqrt{3}i$ (A)

3. $3x^2 + (2-i)x + (4+i) = 0$

$\alpha + \beta = \frac{-b}{a}$

$= \frac{-2+i}{3}$

$\alpha\beta = \frac{c}{a}$

$= \frac{4+i}{3}$

$\frac{1}{z} + \frac{1}{w} = \frac{w+z}{wz}$

$= \dfrac{\frac{-2+i}{3}}{\frac{4+i}{3}}$

$= \frac{-2+i}{4+i}$

$= \frac{-2+i}{4+i} \times \frac{4-i}{4-i}$

$= \frac{-8+6i-i^2}{17}$

$= \frac{-7+6i}{17}$

$\therefore \frac{1}{\bar{z}} + \frac{1}{\bar{w}} = \frac{-7-6i}{17}$ (A)

Question 2

a) $z = 3+i$ $w = 2-5i$

(i) $z^2 = (3+i)^2$

$= 9+6i+i^2$

$= 8+6i$

(ii) $\bar{z} + w = 3-i+2-5i$

$= 5-6i$

(iii) $\frac{w}{z} = \frac{2-5i}{3+i}$

$= \frac{2-5i}{3+i} \times \frac{3-i}{3-i}$

$= \frac{6-17i+5i^2}{10}$

$= \frac{1}{10} - \frac{17}{10}i$

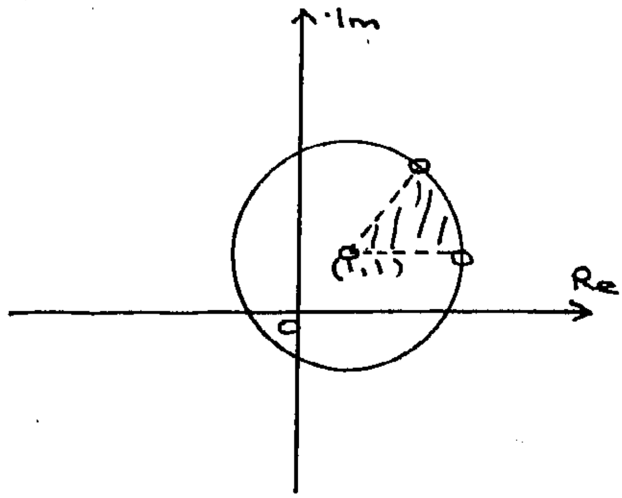
b) $(x+iy)^2 = 24+10i$

$x^2 - y^2 = 24$

$xy = 5$

$\therefore$ The 2 square roots are $\pm(5+i)$

c) $|z-1-i| \le 2$ $0 < \arg(z-1-i) < \frac{\pi}{4}$



d) (i) $z = 1+i$

$|z| = \sqrt{1^2+1^2}$ $\qquad$ $\arg z = \tan^{-1} 1 \quad -\pi < \theta \leq \pi$

$= \sqrt{2}$ $\qquad$ $= \frac{\pi}{4}$

$\therefore 1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$(1+i)^{17} = (\sqrt{2})^{17}\left(\cos\frac{17\pi}{4} + i\sin\frac{17\pi}{4}\right)$

$= 256\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= 256 + 256i$

e) Show $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \quad n \geq 1$

Step 1 Test $n=1$

L.H.S $= (\cos\theta + i\sin\theta)^1$

$= \cos\theta + i\sin\theta$

R.H.S $= \cos\theta + i\sin\theta$

L.H.S = R.H.S

$\therefore$ Result is true for $n=1$

Step 2. Assume that the result is true for $n=k$

that is $(\cos\theta + i\sin\theta)^k = \cos(k\theta) + i\sin(k\theta)$

Step 3 Hence show the result is true for $n=k+1$

that is $(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$

L.H.S $= (\cos\theta + i\sin\theta)^{k+1}$

$= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$

$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$ by step 2

$= \cos k\theta\cos\theta + i\sin\theta\cos k\theta + i\sin k\theta\cos\theta + i^2\sin\theta\sin k\theta$

$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\cos k\theta\sin\theta + \sin k\theta\cos\theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$

= R.H.S

Step 4 Since the result is true for $n=1$ then from Step 3 it is true for $n=1+1=2$, and then for $n=3$, and so on by the process of mathematical induction it is true for all positive integral values of n

e) $\omega^3 = 1$

$\omega^3 - 1 = 0$

$(\omega-1)(\omega^2+\omega+1) = 0$

$\omega - 1 \neq 0 \quad \therefore \quad \omega^2+\omega+1 = 0$

(i) $(1+2\omega+3\omega^2)(1+3\omega+2\omega^2)$

$= 1+3\omega+2\omega^2+2\omega+6\omega^2+4\omega^3+3\omega^2+9\omega^3+6\omega^4$

$= 1+5\omega+11\omega^2+13\omega^3+6\omega^4$

$= 1+5\omega+11\omega^2+13+6\omega$

$= 14+11\omega+11\omega^2$

$= 14+11(\omega^2+\omega)$

$= 14-11$

$= 3$

NB ① $\omega^3 = 1$

② $\omega^4 = \omega$

③ $\omega^2+\omega+1 = 0$

$\omega^2+\omega = -1$

(ii) $1+2\omega+3\omega^2+1+3\omega+2\omega^2 = 2+5\omega+5\omega^2$

$= 2+5(\omega+\omega^2)$

$= 2-5$

$= -3$

(iii) From parts (i) and (ii) $(1+2\omega+3\omega^2)$ and $(1+3\omega+2\omega^2)$ are roots of $x^2+3x+3=0$

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-3\pm\sqrt{3^2-4\times1\times3}}{2\times1}$$

$$= \frac{-3\pm i\sqrt{3}}{2}$$

$\therefore \; 1+2\omega+3\omega^2 = \dfrac{-3-i\sqrt{3}}{2}$

$1+3\omega+2\omega^2 = \dfrac{-3+i\sqrt{3}}{2}$

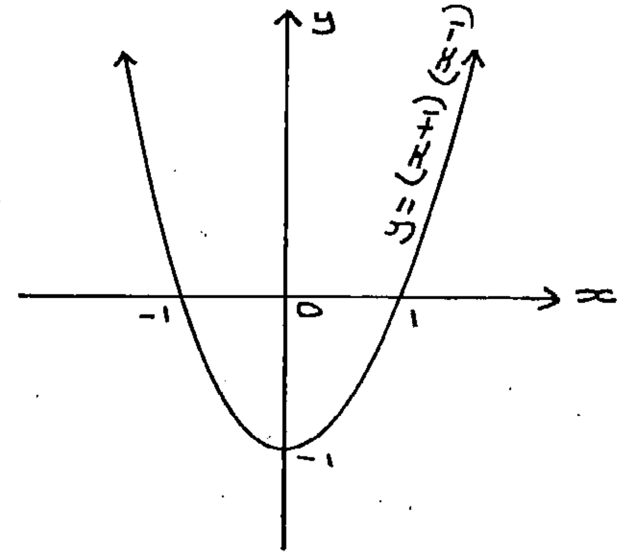
NB $\operatorname{Im}(\omega) = -\operatorname{Im}(\omega^2)$

$\operatorname{Im}(2\omega+3\omega^2) = \operatorname{Im}(\omega^2) < 0$

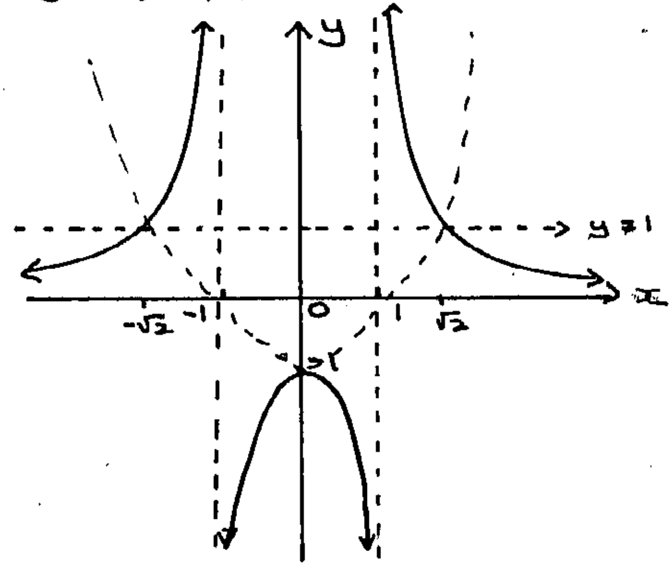
$\operatorname{Im}(3\omega+2\omega^2) = \operatorname{Im}\omega > 0$

Question 3

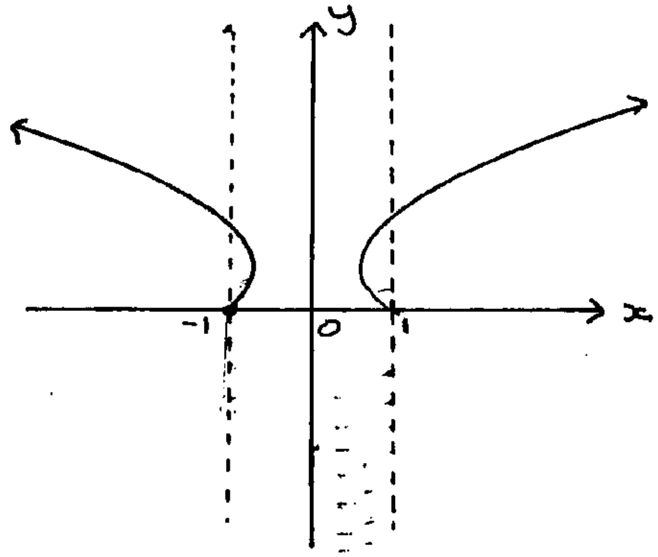
a) (i) $y = (x+1)(x-1)$



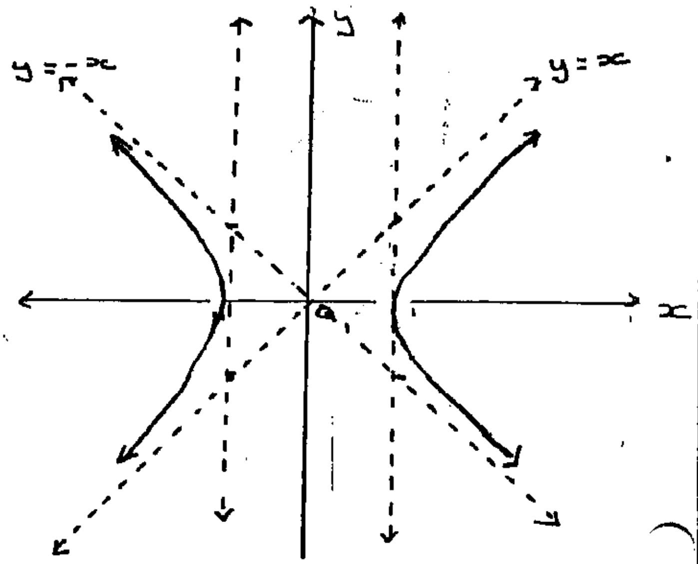
ii) $y = \frac{1}{(x-1)(x+1)}$



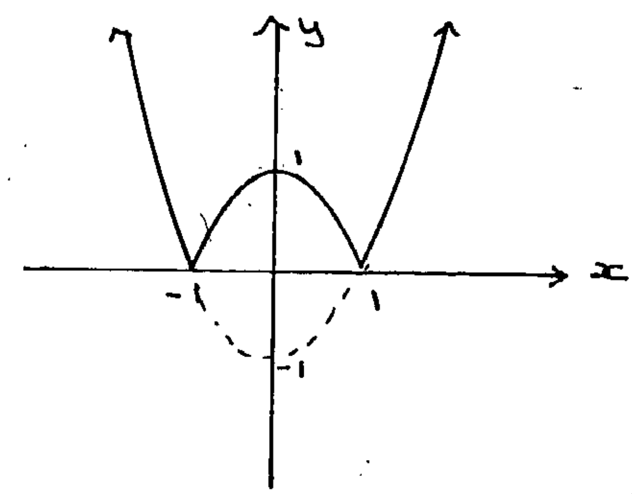
(iii) $y = \sqrt{(x-1)(x+1)}$



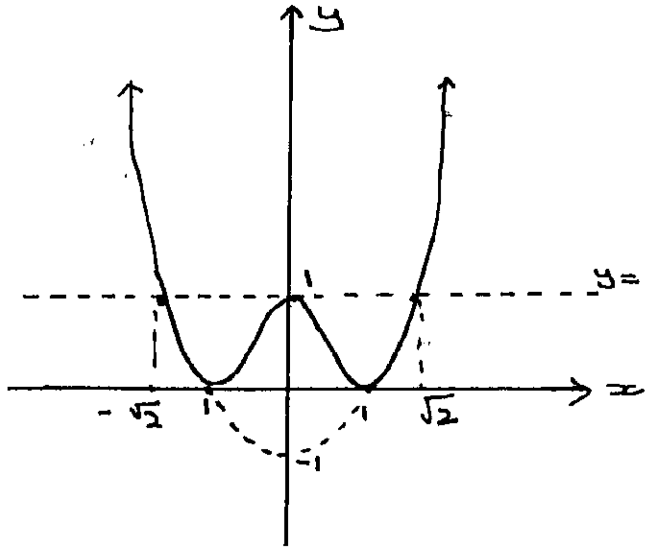
(iv) $y^2 = (x+1)(x-1)$



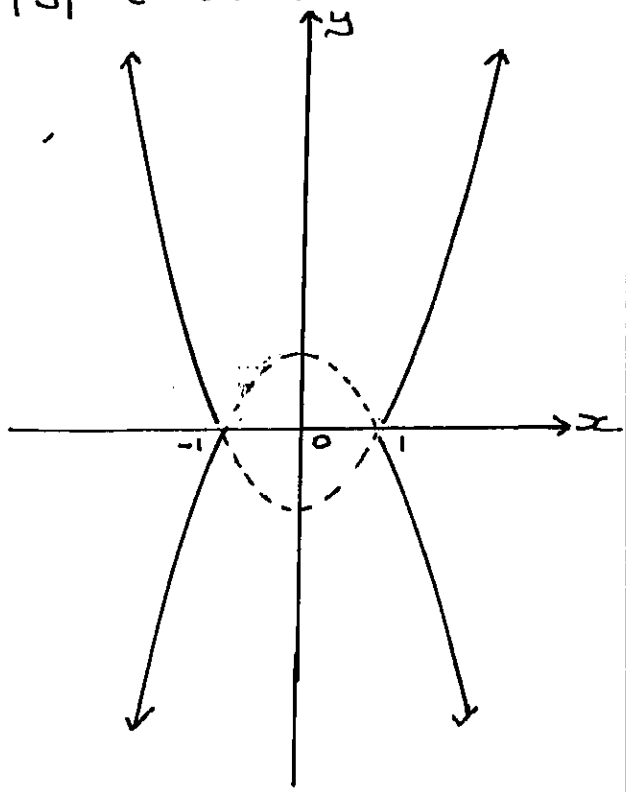
(v) $y = |(x-1)(x+1)|$



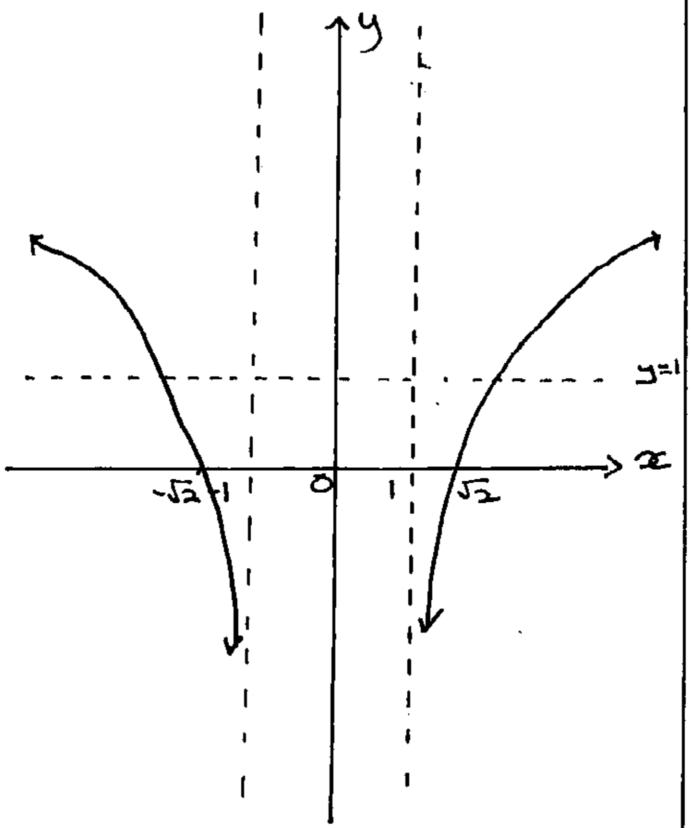
(vi) $y = [(x+1)(x-1)]^2$



vii) $|y| = (x-1)(x+1)$



viii) $y = \log_e (x+1)(x-1)$



b) $f(x) = \dfrac{x^4}{x^2-1}$

$f(-x) = \dfrac{(-x)^4}{(-x)^2-1}$

$= \dfrac{x^4}{x^2-1}$

$\therefore f(x) = f(-x)$

$\therefore f(x) = \dfrac{x^4}{x^2-1}$ is an even function.

ii) $f(x) = \dfrac{x^4}{x^2-1}$

$u = x^4 \qquad v = x^2-1$

$\dfrac{du}{dx} = 4x^3 \qquad \dfrac{dv}{dx} = 2x$

$$f'(x) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(x^2-1)4x^3 - x^4 \times 2x}{(x^2-1)^2}$$

$$= \frac{2x^3(2x^2-2-x^2)}{(x^2-1)^2}$$

$$= \frac{2x^3(x^2-2)}{(x^2-1)^2}$$

$$= \frac{2x^3(x-\sqrt{2})(x+\sqrt{2})}{(x^2-1)^2}$$

For stationary points

$f'(x) = 0$

$\therefore 2x^3(x-\sqrt{2})(x+\sqrt{2}) = 0$

$\therefore x = 0, \pm\sqrt{2}$

$$f(x) = \frac{x^4}{x^2-1}$$

$$f(0) = 0$$

$$f(\sqrt{2}) = \frac{(\sqrt{2})^4}{(\sqrt{2})^2-1}$$

$$= 4$$

$$f(-\sqrt{2}) = \frac{(-\sqrt{2})^4}{(-\sqrt{2})^2-1}$$

$$= 4$$

$\therefore$ The stationary points are $(0,0)$ $(\sqrt{2},4)$ and $(-\sqrt{2},4)$

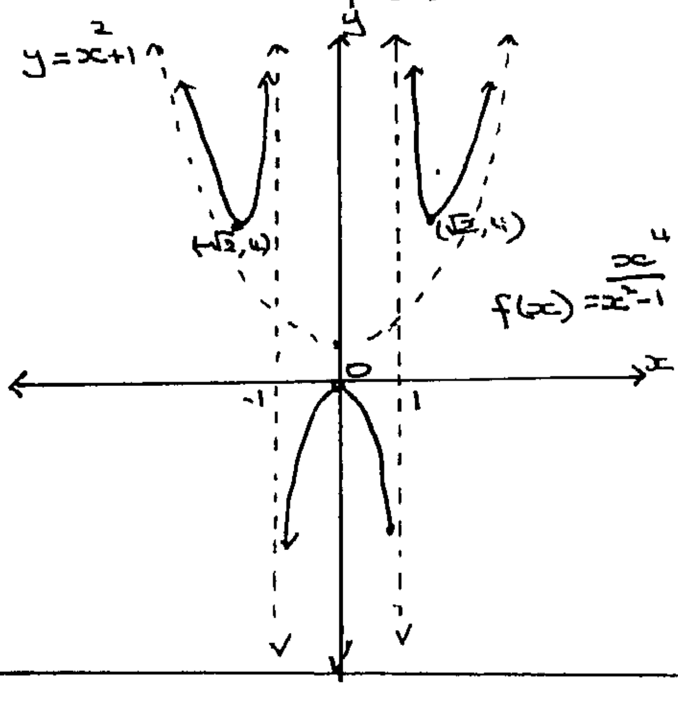
(iii) vertical asymptotes are $x = \pm 1$

$$f(x) = \frac{x^4}{x^2-1}$$

$$= x^2+1 + \frac{1}{x^2-1}$$

as $x \longrightarrow \infty$ $f(x) \longrightarrow x^2+1$

as $x \longrightarrow -\infty$ $f(x) \longrightarrow x^2+1$



c) $3x^2+y^2-2xy-8x+2=0$ — ①

$$6x+2y\frac{dy}{dx}-\left(2x\frac{dy}{dx}+2y\right)-8=0$$

$$(2y-2x)\frac{dy}{dx} = 8+2y-6x$$

$$\frac{dy}{dx} = \frac{8+2y-6x}{2y-2x}$$

$$\therefore \frac{dy}{dx} = \frac{4+y-3x}{y-x}$$

(ii) As the tangent is parallel to $y=2x$, $\frac{dy}{dx}=2$

$$\frac{4+y-3x}{y-x} = 2$$

$$4+y-3x = 2y-2x$$

$$y = 4-x \quad \text{— ②}$$

sub ② in ①

$$3x^2+y^2-2xy-8x+2=0$$

$$3x^2+(4-x)^2-2x(4-x)-8x+2=0$$

$$3x^2+16-8x+x^2-8x+2x^2-8x+2=0$$

$$6x^2-24x+18=0$$

$$x^2-4x+3=0$$

$$(x-3)(x-1)=0$$

$$x = 3, 1$$

when $x = 3$

sub ② $y = 4-x$

$= 4-3$

$= 1$

when $x=1$

sub ② $y = 4-x$

$= 4-1$

$= 3$

$\therefore$ The points are $(3,1)$ $(1,3)$

Question 4

1. $P(x) = x^4 - 1$

$= (x^2-1)(x^2+1)$

$= (x-1)(x+1)(x-i)(x+i)$

(C)

2. $x^3 - 8x^2 - 4x + 12 = 0$

$\alpha+\beta+\gamma = \frac{-b}{a}$

$= 8$

$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a}$

$= -4$

$\alpha\beta\gamma = \frac{-d}{a}$

$= -12$

$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma} = \frac{\alpha\beta+\alpha\gamma+\beta\gamma}{\alpha\beta\gamma}$

$= \frac{-4}{-12}$

$= \frac{1}{3}$ (C)

3. $\frac{1}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$

$1 = A(x-3) + B(x+1)$

sub $x=3$

$1 = B(3+1)$

$B = \frac{1}{4}$

sub $x=-1$

$1 = A(-1-3)$

$A = -\frac{1}{4}$

$\therefore A = -\frac{1}{4}, \; B = \frac{1}{4}$ (D)

Question 5

a) $x^3 - 4x^2 + 5x + 2 = 0$

$\alpha+\beta+\gamma = \frac{-b}{a}$

$= 4$

$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a}$

$= 5$

$\alpha\beta\gamma = \frac{-d}{a}$

$= -2$

(i) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= 4^2 - 2\times5$

$= 6$

(ii) $\alpha^3 = 4\alpha^2 - 5\alpha - 2$ — ①

$\beta^3 = 4\beta^2 - 5\beta - 2$ — ②

$\gamma^3 = 4\gamma^2 - 5\gamma - 2$ — ③

① + ② + ③

$\alpha^3+\beta^3+\gamma^3 = 4(\alpha^2+\beta^2+\gamma^2) - 5(\alpha+\beta+\gamma) - 2\times3$

$= 4(6) - 5(4) - 6$

$= -2$

b) $3x^4 + 17x^3 + 30x^2 + 12x - 8 = 0$

$P(x) = 3x^4 + 17x^3 + 30x^2 + 12x - 8$

$P'(x) = 12x^3 + 51x^2 + 60x + 12$

$P''(x) = 36x^2 + 102x + 60$

$= 2(18x^2 + 51x + 30)$

$= 2(x+2)(18x+15)$

$P(-2) = 0$

$P'(-2) = 0$

$P''(-2) = 0$

$\therefore (x+2)$ is a triple root

$\therefore 3x^4 + 17x^3 + 30x^2 + 12x - 8 = 0$

$(x+2)^3(3x-1) = 0$

by inspection

c) By De Moivre's Theorem $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

i) $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

$$\begin{aligned} L.H.S &= (\cos\theta + i\sin\theta)^3 \\ &= \cos^3\theta + 3\cos^2\theta\, i\sin\theta + 3\cos\theta(i\sin\theta)^2 + i^3\sin^3\theta \\ &= \cos^3\theta + 3\cos^2\theta\, i\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \\ &= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta) \end{aligned}$$

Equating real parts

$$\begin{aligned} \cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\ &= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) \\ &= \cos^3\theta - 3\cos\theta + 3\cos^3\theta \end{aligned}$$

$\therefore \cos 3\theta = 4\cos^3\theta - 3\cos\theta$

ii) $8x^3 - 6x - 1 = 0$

$4x^3 - 3x = \frac{1}{2}$

But $\cos 3\theta = \frac{1}{2}$

$3\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

$\theta = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$

<u>NB</u> $3\theta = \pm\frac{\pi}{3} + 2n\pi$
n is integral

$\therefore$ The roots of $8x^3 - 6x - 1 = 0$ are $\cos\frac{\pi}{9}$, $\cos\frac{5\pi}{9} = -\cos\frac{4\pi}{9}$ and $\cos\frac{7\pi}{9} = -\cos\frac{2\pi}{9}$

(iii) $8x^3 - 6x - 1 = 0$

product of the roots $= \frac{-d}{a}$

$= \frac{1}{8}$

$\therefore \cos\frac{\pi}{9} \times -\cos\frac{4\pi}{9} \times -\cos\frac{2\pi}{9} = \frac{1}{8}$

$\therefore \cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}$

d) $$\frac{2x+31}{(x-1)^3(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{(x+2)}$$

$$2x+31 = A(x-1)^2(x+2) + B(x-1)(x+2) + C(x+2) + D(x-1)^3$$

Let $x=1$ $\quad 2(1)+31 = C(1+2)$

$$33 = 3C$$

$$C = 11$$

Let $x=-2$ $\quad 2(-2)+31 = D(-2-1)^3$

$$27 = -27D$$

$$D = -1$$

Let $x=0$ $\quad 2(0)+31 = A(-1)^2(2) + B(-1)(2) + 11(2) - 1(-1)^3$

$$31 = 2A - 2B + 22 + 1$$

$$8 = 2A - 2B$$

$$4 = A - B \quad \text{—— } ①$$

Let $x=-1$ $\quad 2(-1)+31 = A(-1-1)^2 + B(-1-1)(-1+2) + 11(-1+2) - 1(-1-1)^3$

$$29 = 4A - 2B + 11 + 8$$

$$10 = 4A - 2B$$

$$5 = 2A - B \quad \text{—— } ②$$

Solve ① and ② simultaneously

$$A - B = 4 \quad \text{—— } ①$$

$$2A - B = 5 \quad \text{—— } ②$$

① − ② $\quad -A = -1$

$$A = 1$$

sub ① $\quad A - B = 4$

$$1 - B = 4$$

$$B = -3$$

$$\therefore A = 1, \quad B = -3, \quad C = 11, \quad D = -1$$

e) $x^3 + 3px + q = 0$

$\alpha + \beta + \gamma = \frac{-b}{a}$

$= 0$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

$= 3p$

$\alpha\beta\gamma = \frac{-d}{a}$

$= -q$

sum of the roots $= \frac{\alpha\beta}{\gamma} + \frac{\beta\gamma}{\alpha} + \frac{\alpha\gamma}{\beta}$

$= \frac{\alpha^2\beta^2 + \beta^2\gamma^2 + \alpha^2\gamma^2}{\alpha\beta\gamma}$

$= \frac{(\alpha\beta + \beta\gamma + \alpha\gamma)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma)}{\alpha\beta\gamma}$

$= \frac{(3p)^2 - q(0)}{-q}$

$= \frac{9p^2}{-q}$

sum of the roots 2 at a time $= \frac{\alpha\beta}{\gamma} \times \frac{\beta\gamma}{\alpha} + \frac{\beta\gamma}{\alpha} \times \frac{\gamma\alpha}{\beta} + \frac{\gamma\alpha}{\beta} \times \frac{\alpha\beta}{\gamma}$

$= \beta^2 + \gamma^2 + \alpha^2$

$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= 0 - 2(3p)$

$= -6p$

Product of the roots $= \frac{\alpha\beta}{\gamma} \times \frac{\alpha\gamma}{\beta} \times \frac{\beta\gamma}{\alpha}$

$= -q$

$\therefore$ The monic equation is $x^3 + \frac{9p^2}{q}x^2 - 6px + q = 0$

(ii) If $\gamma = \alpha\beta$ then one of the roots, $\frac{\alpha\beta}{\gamma} = \frac{\gamma}{\gamma} = 1$

$$\therefore \quad x^3 + \frac{9p^2}{q}x^2 - 6px + q = 0$$

$$1 + \frac{9p^2}{q} - 6p + q = 0$$

$$q + 9p^2 - 6pq + q^2 = 0$$

$$q + (3p-q)(3p-q) = 0$$

$$\therefore \quad (3p-q)^2 + q = 0$$